Graphs, part 2

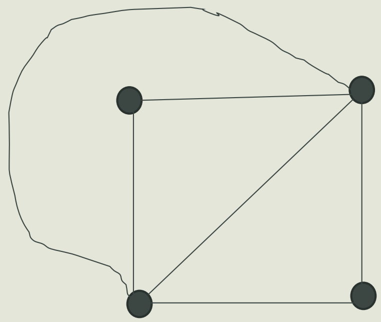
Lecture 8 Feb 28, 2021

Warm up Quiz

- How many (simple) graphs are there with just three vertices?
- Recall: simple means no loops, and at most one edge between every two nodes

Planar graphs; review

- **Planar graph:** if it can be drawn so that no two edge intersect
- **Euler Formula**: $v - e + f = 2$



- **Lemma:** If G is a connected planar graph and $v > 2$ than $e < 3v - 6$

Proof: We may assume $e > 3$.

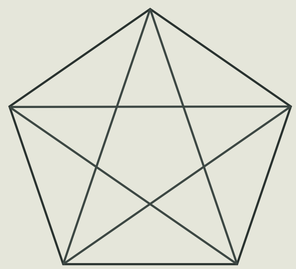
Since there are at least 3 edges, every face has at least 3 edges bounding it.

Furthermore, every edge bounds at most 2 faces. Thus, $3f \leq 2e$.

Euler Formula implies: e $= v + f - 2 \leq v + \frac{2}{3}e - 2 \rightarrow \quad e \leq 3v - 6$

Question

- **Q.** The following graph can not be drawn planar

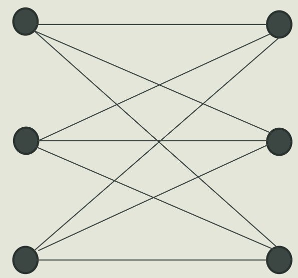


- $e = 10$
- $3v-6= 9$
- But 10 is not less than 9 !

Remark. This graph is known as K5: Complete graph with 5 vertices.

Question

- **K3,3 graph**: complete bipartite graph with 3 nodes on each side
- Can K3,3 be drawn planar?



- The same trick does not work: $e = 9 \quad v = 6$
- $9 < 3v - 6 = 12$ (so this is OK)

- **Lemma**. If G is planar and every cycle has length $\geq L$, then

$$e \leq \frac{L}{L-2}(v-2)$$

Every face has at least L edges bounding it.

Furthermore, every edge bounds at most 2 faces. Thus, $\mathrm{L}\, f \leq 2\, e$.

Euler Formula implies: $\mathrm{e} = v + f - 2 \leq v + \frac{2}{L} e - 2 \rightarrow \quad e \leq \frac{L}{L-2}(v - 2)$

- Now Apply This to K3,3

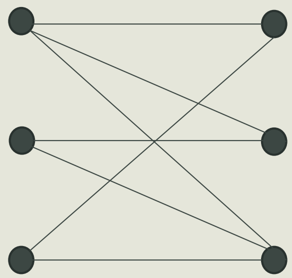
Quiz

- Are trees planar?
- If Yes, verify that Euler Formula holds

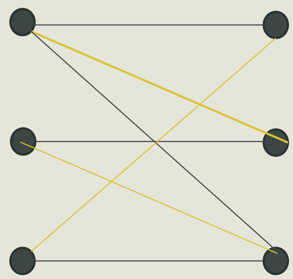
More on bipartite graphs

Matchings and Perfect Matchings:

- A matching is a subset $M \subset E$ of edges such that each node $v \in V$ belongs to at most one edge $e \in M$



- A perfect matching is a subset $M \subset E$ of edges such that each $v \in V$ belongs to exactly one edge $e \in M$ (Picture Right-Below)
- Observation: In order to have a perfect matching in a bipartite graph we must have the same number of nodes on both sides



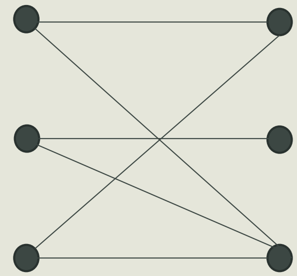
- Question: We saw a necessary condition above for existence of a perfect matching. Is there a necessary and sufficient condition?

Hall's Marriage Theorem

- Let G be a bipartite graph. Denote the set of vertices on Left and Right by V_L and V_R. Suppose $|V_L| = |V_R|$ and that for every $S \subseteq V_L$, the subset $\Gamma(S) \subset V_R$ of vertices in V_R connected to S satisfies

$$|\Gamma(S)| \geq |S|$$

Then G has a perfect matching.



- Proof: We use induction on size of E and some case by case analysis

- **Case 1:** If for every $S \subseteq V_L$, with $0 < |S| < |V_L|$, we have $|\Gamma(S)| \geq |S|$+1

Then

(1) Choose any edge e, (2) Put e in M, (3) Remove e and its end nodes,

(4) We end up with a smaller graph that still satisfies the condition

- **Case 2:** For some $S \subseteq V_L$, with $0 < |S| < |V_L|$, we have $|\Gamma(S)| = |S|$+1

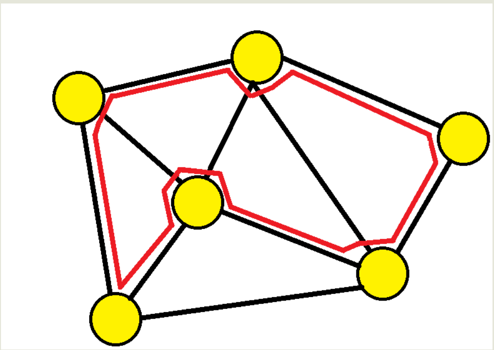
Apply induction to the subgraph with S on left and $\Gamma(S)$ on right to get a perfect match between S and $\Gamma(S)$

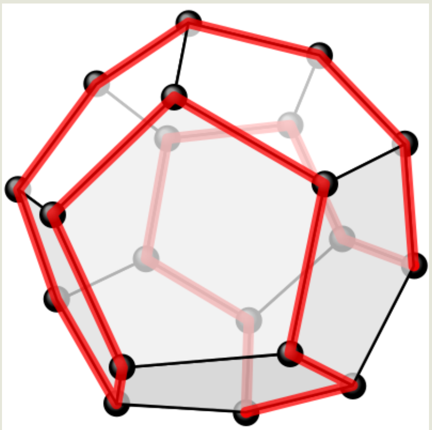
Remove S and $\Gamma(S)$, check that the rest satisfies the condition, and apply induction to the rest

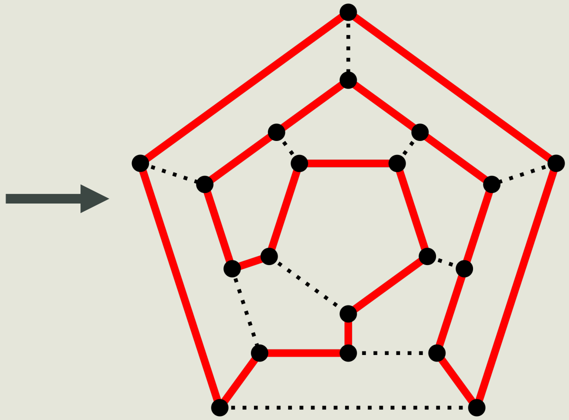
Hamiltonian paths

- Recall: An Eulerian **path/cycle** is a path/cycle that visits each edge exactly once
- A **Hamiltonian path/cycle** is a path/cycle that visits each vertex exactly once

(images courtesy of Wikipedia)





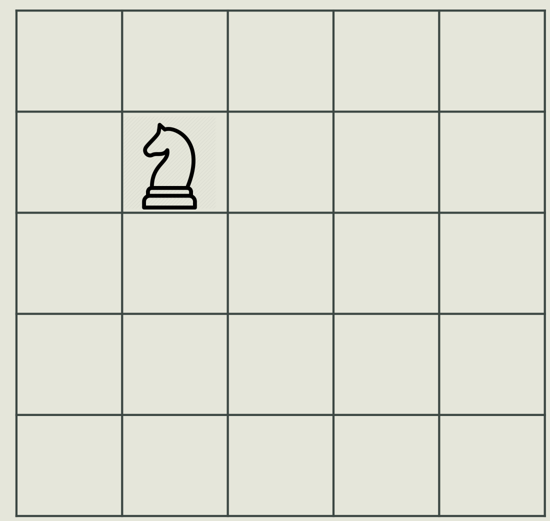


Quiz

- Which trees have a Hamiltonian path?
- Which trees have an Eulerian path?

Knight move problems

- **Question.** Is it possible for a Knight to go around a 5x5 chessboard from a given square visiting each square once and only once (and return to the original square).



- Wiki Page: https://en.wikipedia.org/wiki/Knight%27s_tour

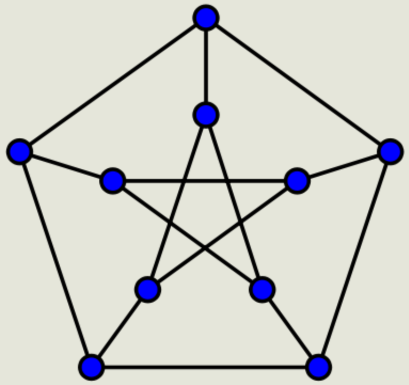
Quiz

- Find the number of Hamilton cycles in
- K4:
- K5:
- K6:
- K7

Petersen Graph

- **Importance.** It is a relatively small graph that provides a counterexample to many important questions

- **Lemma.** Petersen graph does not have a Hamilton cycle.



Links for reading

- Important Problems in Graph Theory: https://towardsdatascience.com/common-graph-theory-problems-ca990c6865f1
- Random Questions to think about:

http://www.geometer.org/mathcircles/graphprobs.pdf

- A free book: https://amsi.org.au/ESA_Senior_Years/PDF/PDFvcaa/graphtheory7a.pdf
- https://sites.math.rutgers.edu/~sk1233/courses/graphtheory-F11/